Value Ranges of Univalent Self-Mappings of the Unit Disc

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September 18, 2015

Abstract

We describe the value set $\{f(z_0) : f : \mathbb{D} \to \mathbb{D} \text{ univalent}, f(0) = 0, f'(0) = e^{-T}\}$, where $\mathbb{D}$ denotes the unit disc and $z_0 \in \mathbb{D} \setminus \{0\}$, $T > 0$, by applying Pontryagin's maximum principle to the radial Loewner equation.

Keywords: Pontryagin's maximum principle, radial Loewner equation, univalent functions
2010 Mathematics Subject Classification: 30C55, 30C80.

1 Introduction and main result

Given a bounded univalent function f on a simply connected domain $\Omega \subsetneq \mathbb{C}$ and two distinct points $a, b \in \Omega$, it is quite natural to ask the question which values $f(b)$ can take if $f(a)$ and $f'(a)$ are prescribed. Since the Riemann mapping theorem tells us that any such domain Ω can be mapped conformally onto the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that a is mapped to 0, the problem can be restricted to the case of $\Omega = \mathbb{D}$ and $a = 0$.

By multiplying with a real constant ≤ 1 and applying an automorphism of $\mathbb{D}$, we may assume $f : \mathbb{D} \to \mathbb{D}$ and $f(0) = 0$. Then the Schwarz lemma tells us that $|f'(0)| \leq 1$ and $|f'(0)| = 1$ if and only if f is the rotation $f(z) = f'(0)z$. In order to describe the non-trivial case $|f'(0)| < 1$, we can restrict ourselves to the case $f'(0) \in (0, 1)$ because of rotational symmetry. Thus we consider the set

$$\mathcal{S}_T := \{f : \mathbb{D} \to \mathbb{D} \text{ univalent}, f(0) = 0, f'(0) = e^{-T}\}, \quad T > 0.$$

In this note, we will determine the value set

$$V_T(z_0) = \{f(z_0) : f \in \mathcal{S}_T\}, \quad z_0 \in \mathbb{D} \setminus \{0\}.$$

Variations of the set $V_T(z_0) = \{f(z_0) : f \in \mathcal{S}_T\}$ have been determined by various authors, from the classical setting of the Schwarz and Rogosinksi's lemma [Rog34], which concern itself with holomorphic functions $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$ that fulfil no further conditions, to a recent paper by Roth and Schleißinger [RS14] that determines the set $\mathcal{V}(z_0) = \{f(z_0) : f \in \mathcal{S}\}$, with the class $\mathcal{S} := \{f : \mathbb{D} \to \mathbb{D} \text{ univalent}, f(0) = 0, f'(0) > 0\}$. Note that $\mathcal{V}(z_0) = \cup_{T>0} V_T(z_0)$.

Our results are analogous to the results of Prokhorov and Samsonova [PS15], who study univalent self-mappings of the upper half-plane having the so called hydrodynamical normalization at the boundary point ∞. Finally we note that in [GG76], the authors consider the set $\{\log(f(z_0)/z_0) : f : \mathbb{D} \to \mathbb{C} \text{ univalent}, \ f(0) = 0, \ |f(z)| \leq M\}$ for $M > 0$. We use a different and more straightforward approach to directly determine the set $V_T(z_0)$ by applying Pontryagin's maximum principle to the radial Loewner equation.

In the following, for the sake of simplicity, we assume that $z_0 \in (0, 1)$; for other values of z_0, we just consider the function $z \mapsto e^{i \arg z_0} f\left(e^{-i \arg z_0} z\right)$ instead of f.

*Supported by the ERC grant "HEVO - Holomorphic Evolution Equations" n. 277691.

Theorem 1.1. *Let $z_0 \in (0,1)$. For $x_0 \in [-1,1]$ and $T > 0$, let $r = r(T, x_0)$ be the (unique) solution to the equation*

$$(1+x_0)(1-z_0)^2 \log(1-r) + (1-x_0)(1+z_0)^2 \log(1+r) - (1 - 2x_0 z_0 + z_0^2)\log r =$$
$$(1+x_0)(1-z_0)^2 \log(1-z_0) + (1-x_0)(1+z_0)^2 \log(1+z_0) - (1-2x_0 z_0 + z_0^2)\log e^{-T} z_0$$

and let

$$\sigma(T, x_0) = \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1 - 2x_0 z_0 + z_0^2}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T, x_0)\right).$$

Furthermore, for fixed $T \geq 0$, define the two curves $C_+(z_0)$ and $C_-(z_0)$ by

$$C_\pm(z_0) := \left\{ w_\pm(x_0) := r(T, x_0) e^{\pm i\sigma(T, x_0)} \; : \; x_0 \in [-1,1] \right\}.$$

Then, if $\operatorname{arctanh} z_0 < \frac{\pi}{2}$, *$V_T(z_0)$ is the closed region whose boundary consists of the two curves $C_+(z_0)$ and $C_-(z_0)$, which only intersect at $x_0 \in \{-1, 1\}$.*

For $\operatorname{arctanh} z_0 \geq \frac{\pi}{2}$, *there are two different cases: First assume that T is large enough that the equation*

$$\frac{2(1-z_0^2)\sqrt{1-x^2}}{1 + 2x z_0 + z_0^2}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T, x)\right) = \pi \tag{1.1}$$

admits a solution $x \in [-1,1]$. Then the curves $C_+(z_0)$ and $C_-(z_0)$ intersect more than twice. There is a $\chi \in (-1,1)$ such that $\widetilde{C}_+(z_0) \cup \widetilde{C}_-(z_0)$ is a closed Jordan curve, where

$$\widetilde{C}_\pm(z_0) := \{ w_\pm(x_0) \; : \; x_0 \in [\chi, 1] \},$$

and an $\aleph \in (-1,1)$ such that $\widehat{C}_+(z_0) \cup \widehat{C}_-(z_0)$ is a closed Jordan curve, where

$$\widehat{C}_\pm(z_0) := \{ w_\pm(x_0) \; : \; x_0 \in [-1, \aleph] \}.$$

Then $V_T(z_0)$ is the closed region whose boundary is $\widetilde{C}_+(z_0) \cup \widetilde{C}_-(z_0) \cup \widehat{C}_+(z_0) \cup \widehat{C}_-(z_0)$.
For smaller T that do not admit a solution to (1.1), the set $V_T(z_0)$ can be described exactly as in the case of $\operatorname{arctanh} z_0 < \frac{\pi}{2}$.

The following figures show the evolution of the sets $V_T(z_0)$ over time. Note that $\operatorname{arctanh} z_0 = \frac{\pi}{2} \iff z_0 = \tanh(\pi/2) \approx 0.917$.

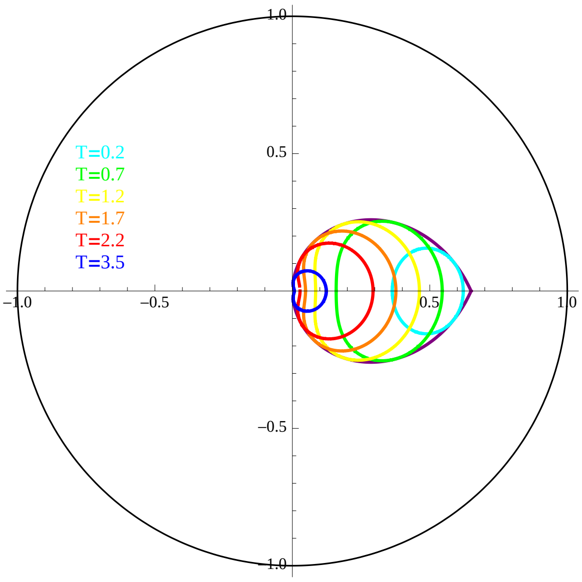

Figure 1: $V_T(0.65)$

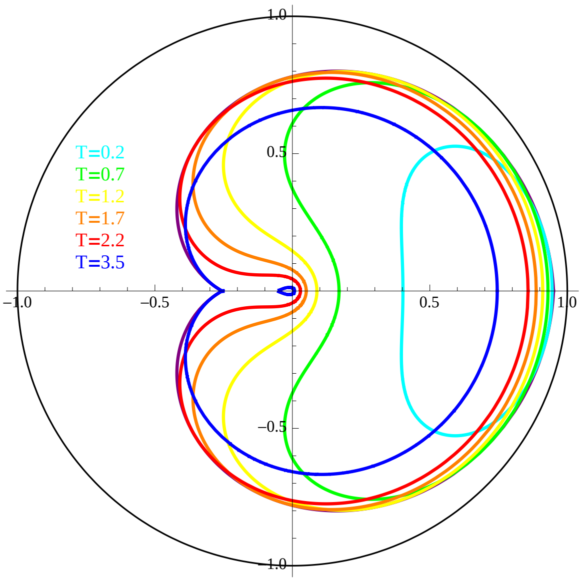

Figure 2: $V_T(0.95)$

The sets $V_T(z_0)$ for $z_0 = 0.65, 0.95$ and $T = 0.2 + 0.5j$, $j = 0, 1, \ldots, 4$, and $T = 3.5$.

We prove Theorem 1.1 in Section 2, and in Section 3 we consider the similar problem of describing the value set $\{f^{-1}(z_0) : f \in \mathcal{S}_T \text{ with } z_0 \in f(\mathbb{D})\}$ for the inverse functions.

2 Proof of Theorem 1.1

Consider the radial Loewner equation

$$\dot{f}_t(z) = -f_t(z) \cdot p(t, f_t(z)) \text{ for a.e. } t \geq 0, \quad f_0(z) = z \in \mathbb{D}, \tag{2.1}$$

where $p : [0, \infty) \times \mathbb{D} \to \mathbb{C}$ is a *Herglotz function*, i.e. for almost every $t \geq 0$, $z \mapsto p(t, z)$ is a holomorphic function with Re $p(z) > 0$ for all $z \in \mathbb{D}$ and $p(0) = 1$ and the function $t \mapsto p(t, z)$ is measurable for every $z \in \mathbb{D}$.
For every $f \in \mathcal{S}_T$ there exists a Herglotz function $p(t, z)$ such that the solution $\{f_t\}_{t\geq 0}$ of (2.1) satisfies $f_T = f$; see [Pom75], Chapter 6.

Thus the description of $V_T(z_0)$ can be translated into the control theoretic problem of describing the reachable set $R_T(z_0)$ of the initial value problem

$$\dot{w}(t) = -w(t) \cdot p(t, w(t)), \quad w(0) = z_0 \in \mathbb{D}, \tag{2.2}$$

where $p(t, z)$ runs through the set of all Herglotz functions and

$$R_T(z_0) := \{w(T) \, : \, w : [0, T] \to \mathbb{D} \text{ solves (2.2)}\}.$$

Then we have $V_T(z_0) = R_T(z_0)$ and, obviously, $R_T(z_0)$ is a closed set.

Denote by $\mathcal{P}$ the set of all probability measures on $\partial\mathbb{D}$. Due to the Herglotz representation ([Dur83], Section 1.9) we can write $p(t, z)$ for a. e. $t \geq 0$ as

$$p(t, z) = p_{\mu_t}(z) := \int_{\partial\mathbb{D}} \frac{u + z}{u - z} \, \mu_t(du), \tag{2.3}$$

for some $\mu_t \in \mathcal{P}$.
For $\mu \in \mathcal{P}$, $\lambda \in \mathbb{C}$ and $w \in \mathbb{D}$ we define the Hamiltonian $H(\mu, \lambda, w)$ by

$$H(\mu, \lambda, w) = -\lambda \cdot w \cdot p_\mu(w).$$

Then (2.2) has the form $\dot{w}_t = \frac{\partial}{\partial\lambda} H(\mu_t, \lambda, w(t))$.

Now, if $\{\mu_t\}_{t\geq 0}$ leads to an extremal solution $w(t)$, i.e. $w(T) \in \partial R_T(z_0)$, then $\{\mu_t\}_{t\geq 0}$, $w(t)$ and $\lambda(t)$ satisfy Pontryagin's maximum principle; see [LM86], p.254, Theorem 3. In our setting we choose complex coordinates and a simple calculation shows that the principle stated in [LM86] then has the following form:
Define $\lambda(t)$ as the solution to the adjoint differential equation

$$\dot{\lambda}(t) = -\frac{\partial}{\partial w} H(\mu_t, \lambda(t), w(t)), \tag{2.4}$$

with the initial value condition

$$\lambda(0) = e^{i\beta}, \text{ with } \beta \in [0, 2\pi).$$

Then, for almost every $t \in [0, T]$, we have

$$\text{Re } H(\mu_t, \lambda(t), w(t)) = \max_{\mu \in \mathcal{P}} \text{Re } H(\mu, \lambda(t), w(t)), \tag{2.5}$$

and

$$\text{Re } H(\mu_t, \lambda(t), w(t)) = \textit{const.} \quad \text{for almost all } t \in [0, T].$$

In passing we note that equations such as (2.2), i.e. evolution equations for holomorphic functions, can also be regarded and studied as control systems; see [Rot98].

From (2.3) it is easy to see that Re $H(\mu, \lambda(t), w(t))$ is maximised only for point measures, i.e. when

$$H(\mu, \lambda, w) = -\lambda \cdot w \cdot \frac{u + w}{u - w}$$

for some $u \in \partial\mathbb{D}$. Thus, for almost every $t \geq 0$, $H(\mu_t, \lambda(t), w(t)) = -\lambda(t) \cdot w(t) \cdot \frac{\kappa(t)+w(t)}{\kappa(t)-w(t)}$, where $\kappa : [0, T] \to \partial\mathbb{D}$ is measurable and (2.2), (2.4) become

$$\dot{w}(t) = -w(t) \cdot \frac{\kappa(t) + w(t)}{\kappa(t) - w(t)}, \quad w(0) = z_0 \in \mathbb{D}, \tag{2.6}$$

$$\dot{\lambda}(t) = -\lambda(t) \cdot \frac{w(t)^2 - 2\kappa(t)w(t) - \kappa(t)^2}{(\kappa(t) - w(t))^2}, \quad \lambda(0) = e^{i\beta}. \tag{2.7}$$

We now optimise the Hamiltonian by rewriting

$$\max_{\kappa \in \partial\mathbb{D}} \mathrm{Re}\ -w\lambda \cdot \frac{\kappa + w}{\kappa - w} = \max_{\phi \in \mathbb{R}} \mathrm{Re}\ (-\lambda w(m + re^{i\phi})) = r|\lambda w| - m\mathrm{Re}\ (\lambda w),$$

where

$$m = \frac{1 + |w|^2}{1 - |w|^2}, \qquad r = \frac{2|w|}{1 - |w|^2}, \qquad e^{i\phi} = \frac{w - |w|^2\kappa}{|w|\kappa - w|w|}.$$

The maximum is then obviously taken at

$$\phi = \pi - \arg(\lambda w) \quad \Leftrightarrow \quad \kappa = \frac{w}{|w|} \frac{1 + |w|e^{i\phi}}{e^{i\phi} + |w|} = w\frac{|\lambda| - \overline{\lambda w}}{|\lambda||w|^2 - \overline{\lambda w}}. \tag{2.8}$$

Inserting this into the phase equation (2.6) yields

$$\dot{w} = -w\left(m + re^{i\phi}\right),$$

or, in polar coordinates,

$$\frac{d}{dt}|w| = -|w|(m + r\cos\phi) = -|w|\left(\frac{1 + |w|^2 - 2|w|\cos(\arg\lambda + \arg w)}{1 - |w|^2}\right), \tag{2.9}$$

$$\frac{d}{dt}\arg w = -r\sin\phi = -\frac{2|w|\sin(\arg\lambda + \arg w)}{1 - |w|^2}, \tag{2.10}$$

and the costate equation (2.7) reads

$$\dot{\lambda} = \lambda\left(m + re^{i\phi} + 2|w|\frac{|w| + e^{i\phi}(1 + |w|^2) + |w|e^{2i\phi}}{(1 - |w|^2)^2}\right),$$

which corresponds to

$$\begin{aligned}\frac{d}{dt}|\lambda| &= |\lambda|\left(m + r\cos\phi + 2|w|\frac{|w| + (1 + |w|^2)\cos\phi + |w|\cos 2\phi}{(1 - |w|^2)^2}\right) = \\ &= |\lambda|\frac{1 - |w|^4 + 2|w|^2 - 4|w|\cos(\arg\lambda + \arg w) + 2|w|^2\cos(2\arg\lambda + 2\arg w)}{(1 - |w|^2)^2},\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\arg\lambda &= r\sin\phi + 2|w|\frac{|w|\sin(2\phi) + (1 + |w|^2)\sin\phi}{(1 - |w|^2)^2} = \\ &= \frac{4|w|\sin(\arg\lambda + \arg w) - 2|w|^2\sin(2\arg\lambda + 2\arg w)}{(1 - |w|^2)^2}.\end{aligned} \tag{2.11}$$

Now we introduce the variable

$$x := \cos(\arg\lambda + \arg w),$$

which reduces our system of equations (2.9), (2.10), (2.11) to

$$\frac{d}{dt}|w| = -|w|\left(\frac{1 + |w|^2 - 2|w|x}{1 - |w|^2}\right) \tag{2.12}$$

and

$$\frac{d}{dt}x = -2|w|(1 - x^2)\frac{1 + |w|^2 - 2x|w|}{(1 - |w|^2)^2} = 2\frac{1 - x^2}{1 - |w|^2}\frac{d|w|}{dt} \tag{2.13}$$

with the initial value conditions

$$|w(0)| = z_0, \qquad x(0) = x_0 := \cos\beta. \tag{2.14}$$

For $x_0^2 \neq 1$, separation of variables solves (2.13), (2.14) as

$$x(t) = \Phi^{-1}\left(2\,\mathrm{arctanh}\,|w(t)| - 2\,\mathrm{arctanh}\,z_0\right),$$

where

$$\Phi(y) := \operatorname{arctanh} y - \operatorname{arctanh} x_0,$$

which means

$$\begin{aligned}
x(t) &= \tanh\left(2\operatorname{arctanh}|w(t)| + \operatorname{arctanh} x_0 - 2\operatorname{arctanh} z_0\right) = \\
&= \frac{(1+|w(t)|^2)\left(x_0 - 2z_0 + x_0 z_0^2\right) + 2|w(t)|\left(1 - 2x_0 z_0 + z_0^2\right)}{(1+|w(t)|^2)\left(1 - 2x_0 z_0 + z_0^2\right) + 2|w(t)|\left(x_0 - 2z_0 + x_0 z_0^2\right)} = \\
&= \frac{(1+|w(t)|^2)A + 2|w(t)|B}{(1+|w(t)|^2)B + 2|w(t)|A}
\end{aligned} \tag{2.15}$$

with

$$\begin{aligned}
A &:= x_0 - 2z_0 + x_0 z_0^2, \\
B &:= 1 - 2x_0 z_0 + z_0^2.
\end{aligned}$$

Note that, in fact, the denominator in (2.15) never equals zero for any $x_0 \in [-1,1]$, since we have

$$\begin{aligned}
(1+|w|^2)B + 2|w|A = 0 \quad \Leftrightarrow \quad |w| &= -\frac{A}{B} \pm \frac{\sqrt{A^2 - B^2}}{B} \\
&= -\frac{A}{B} \pm \frac{\sqrt{(x_0^2 - 1)(1 - z_0^2)^2)}}{B},
\end{aligned}$$

which only yields real terms for $x_0^2 = 1$, and in this case the only solution is

$$|w| = -\frac{A}{B} = \pm 1 \notin (0,1).$$

Therefore, (2.15) is for all $x_0 \in [-1,1]$ the solution to the initial value problem (2.13), and thus (2.12) can be simplified to

$$\frac{d}{dt}|w(t)| = -|w| \frac{B(1 - |w|^2)}{B(1 + |w|^2) + 2|w|A}, \quad |w(0)| = z_0.$$

The function

$$\Psi(y) := (A+B)\log(1-y) - B\log(y) - (A-B)\log(1+y)$$

is strictly monotonous on the interval $(0,1)$, since its derivative is zero-free. Hence it is invertible, and

$$|w(t)| = \Psi^{-1}(Bt + \Psi(z_0)),$$

is the solution to the initial value problem (2.12), which can be verified by calculation.
To determine the value set $R_T(z_0)$, we solve the remaining initial value problem (2.10), which now reads

$$\frac{d}{dt}\arg w(t) = \pm \frac{2\sqrt{B^2 - A^2}}{B(1 + |w(t)|^2) + 2A|w(t)|}, \quad \arg w(0) = 0.$$

If we write

$$\arg w(t) = -G(|w(t)|),$$

where G is the solution to

$$\frac{d}{d|w|}G(|w|) = \frac{2\sqrt{B^2 - A^2}}{B(1 - |w|^2)}, \quad G(0) = 0,$$

then

$$\arg w(t) = \frac{\pm 2\sqrt{B^2 - A^2}}{B}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh}|w(t)|\right).$$

We can therefore describe candidates for the boundary points of the set $R_T(z_0)$ as follows:
For $x_0 \in [-1,1]$, let $r = r(T,x_0)$ be the (unique) solution to the equation

$$(1+x_0)(1-z_0)^2\log(1-r) + (1-x_0)(1+z_0)^2\log(1+r) - (1-2x_0z_0+z_0^2)\log r = \\ (1+x_0)(1-z_0)^2\log(1-z_0) + (1-x_0)(1+z_0)^2\log(1+z_0) - (1-2x_0z_0+z_0^2)\log e^{-T}z_0, \quad (2.16)$$

then $\partial R_T(z_0)$ consists of a subset of the two curves

$$C_\pm(z_0) = \left\{ w_\pm(x_0) = r(T,x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0 \in [-1,1] \right\},$$

where

$$\sigma(T,x_0) = \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1-2x_0z_0+z_0^2}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T,x_0)\right).$$

First we consider the function $x_0 \mapsto r(T,x_0)$: By solving (2.16) for T and then taking the derivative with respect to x_0, we obtain

$$\frac{\partial}{\partial x_0} r(T,x_0) = -\frac{(1-z_0)^2 r(T,x_0)(1-r^2(T,x_0))\left(\log\left(\frac{1+r(T,x_0)}{1-r(T,x_0)}\right) - \log\left(\frac{1+z_0}{1-z_0}\right)\right)}{B(B(1+r^2(T,x_0)) + 2A\, r(T,x_0))},$$

and since the only zeros of this term lie at $r(T,x_0) = 0$, $r(T,x_0) = \pm 1$ and $r(T,x_0) = z_0$, this immediately shows that $x_0 \mapsto r(T,x_0)$ is strictly increasing.
In particular, the curves $C_+(z_0)$ and $C_-(z_0)$ do not hit themselves.

Now we consider the first case where $z_0 < \tanh\frac{\pi}{2}$. Here, the curves never hit the negative real axis: As the function

$$x_0 \mapsto \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1-2x_0z_0+z_0^2}$$

reaches its single maximal value 2 at $x_0 = \frac{2z_0}{1+z_0^2}$, we have

$$\sigma(T,x_0) = \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1+2x_0z_0+z_0^2}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T,x_0)\right) < 2\cdot(\pi/2-0) = \pi.$$

Thus, they intersect only on the positive real axis and, as $\sigma(T,x_0) = 0$ if and only if $x_0 = \pm 1$, this happens exactly at $x_0 = \pm 1$. Hence, the full set $C_+(z_0) \cup C_-(z_0)$ forms the boundary of $R_T(z_0)$. Since $R_T(z_0)$ is obviously bounded, it has to consist of the bounded region enclosed by the two curves.

Next assume that $z_0 > \tanh\frac{\pi}{2}$. We have

$$\frac{\partial}{\partial x_0}\sigma(T,x_0) = -\frac{1-z_0^2}{B\sqrt{1-x_0^2}}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T,x_0)\right)\frac{A(1+r(T,x_0)^2) + 2Br(T,x_0)}{B(1+r(T,x_0)^2) + 2Ar(T,x_0)}.$$

The zeros of this term lie clearly at the points $x_0 \neq \frac{2z_0}{1+z_0^2}$ with

$$r(T,x_0) = \frac{-B \pm \sqrt{B^2-A^2}}{A}.$$

Since

$$\frac{-B-\sqrt{B^2-A^2}}{A} \qquad \begin{cases} \geq 1 & \text{for } x_0 < \frac{2z_0}{1+z_0^2} \\ < 0 & \text{for } x_0 > \frac{2z_0}{1+z_0^2}, \end{cases}$$

it is clear that this term can be ignored. We focus on the equality

$$r(T,x_0) = \frac{-B+\sqrt{B^2-A^2}}{A} \quad (2.17)$$

and note that here the term on the right-hand side is well-defined for all $x_0 \in [-1,1]$, and strictly decreasing on this interval, taking values between -1 and 1. Therefore, $x_0 \mapsto h(x_0) := \frac{-B+\sqrt{B^2-A^2}}{A} - r(T,x_0)$ is continuous on $[-1,1]$, strictly decreasing, and we have $h(-1) \geq 0$ and $h(1) \leq 0$. Thus (2.17) has exactly one solution $x_0 = x^*$ on $[-1,1]$, and the function $x_0 \mapsto \sigma(T,x_0)$

increases from 0 to $\sigma(T, x^*)$ and decreases again to 0.

If T is so small that equation (1.1) has no solution, then we are again in the same situation: the two curves intersect only twice, namely for $x_0 = \pm 1$, and $R_T(z_0)$ is the closed region bounded by the two curves.

There is a T^* such that (1.1) admits a solution, but has no solution for any $T < T^*$. At this T^*, the curves $C_\pm(z_0)$ will meet for the first time, i.e. $\sigma(T^*, x^*) = \pi$. This means that at x^*, the curves both touch $\mathbb{R}^-$ at some point z^*, see Figure 3, and $R_T(z_0)$ (shown in green) is no longer simply connected, since the component containing the origin can obviously not be part of $R_T(z_0)$.

For slightly larger T, the curves $C_\pm(z_0)$ intersect on $\mathbb{R}^-$ twice and $\mathbb{D} \setminus (C_+(z_0) \cup C_-(z_0))$ has four components, see Figure 4. We denote by $K_T(z_0)$ the component (shown in orange) that arises from the intersection of the two curves near $x_0 = x^*$. Obviously, the component that contains the origin,

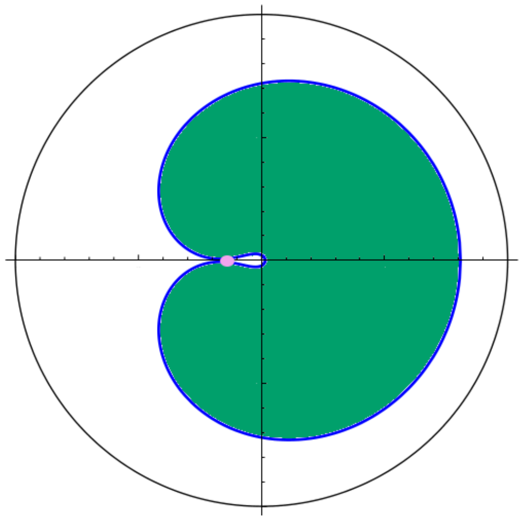

Figure 3: $V_{T^*}(z_0)$

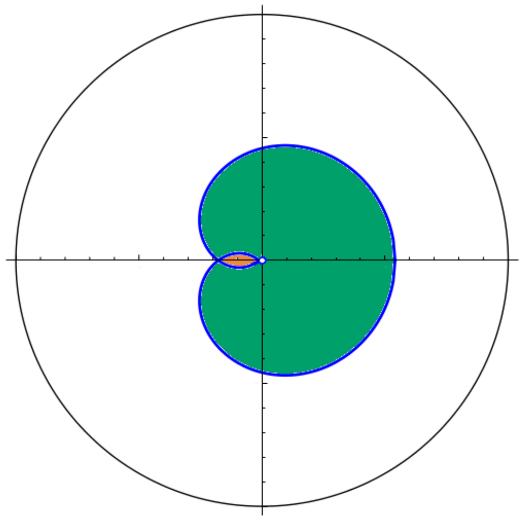

Figure 4: $V_{T^*+\varepsilon}(z_0)$

The evolution of the decomposition of $\mathbb{D}$ by $C_\pm(z_0)$

as well as the "exterior" component (both shown in white) cannot be part of $R_T(z_0)$. For reasons of continuity, the "large interior" component (shown in green) must belong to $R_T(z_0)$. It remains to show that $K_T(z_0)$ also belongs to $R_T(z_0)$:
Since $z^* = w(T^*)$ for a solution $w(t)$ of the Loewner equation (2.6), we know that $R_T(z_0)$ contains the set $R_{T-T^*}(z^*)$, which we determined already if $T - T^*$ is small enough. In particular, $R_T(z_0)$ contains infinitely many points of $\mathbb{R}^-$. If $K_T(z_0)$ was not included in $R_T(z_0)$, then $R_T(z_0) \cap \mathbb{R}^-$ would consist of only two points, a contradiction.
For reasons of continuity, the set $R_T(z_0)$ will have the form described in the theorem for any larger T as well, and this concludes the proof.

Remark 2.1. *If $w_0 \in \partial V_T(z_0)$, then there exists exactly one control function $\kappa(t)$ such that the solution $\{f_t\}_{t\in[0,T]}$ of* (2.1) *with $p(t,z) = \frac{\kappa(t)+z}{\kappa(t)-z}$ satisfies $f_T(z_0) = w_0$. Equation* (2.8) *shows that $\kappa(t) = \exp(i\varphi(t))$ is continuously differentiable. From [MR05], Theorem 1.1, it follows that f is a slit mapping in this case, i.e. f maps $\mathbb{D}$ conformally onto $\mathbb{D} \setminus \gamma$, where γ is a simple curve.*

3 Value sets for the inverse functions

Firstly, in analogy to [RS14] and the set $\mathcal{V}(z_0)$, we describe the set

$$\mathcal{W}(z_0) := \{f^{-1}(z_0) \,:\, f \in \mathcal{S} \text{ with } z_0 \in f(\mathbb{D})\}.$$

In the following we write $d_\mathbb{D}(0,z)$, $z \in \mathbb{D}$, for the hyperbolic distance between 0 and z (using the hyperbolic metric with curvature -1), i.e. $d_\mathbb{D}(0,z) = 2\,\mathrm{arctanh}(|z|) = \log\left(\frac{1+|z|}{1-|z|}\right)$.

Theorem 3.1. *We have*

$$\begin{aligned}\mathcal{W}(z_0) &= \{f^{-1}(z_0) \,:\, f:\mathbb{D}\to\mathbb{D} \text{ univalent},\ f(0)=0,\ f'(0)>0 \text{ with } z_0\in f(\mathbb{D})\}\\ &= \{re^{i\sigma} \,:\, d_{\mathbb{D}}(0,r)\geq|\sigma|+d_{\mathbb{D}}(0,z_0),\, \sigma\in[-\pi,\pi]\}.\end{aligned}$$

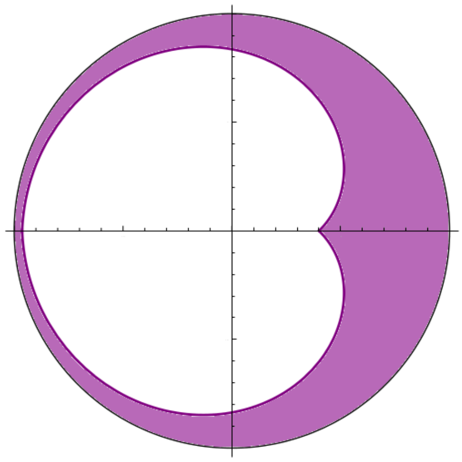

Figure 5: The set $\mathcal{W}(0.4)$

Furthermore, we will determine the value set

$$W_T(z_0) := \{f^{-1}(z_0) \,:\, f\in\mathcal{S}_T \text{ with } z_0\in f(\mathbb{D})\}$$

for the inverse functions:

Theorem 3.2. *Let $z_0\in(0,1)$. For $x_0\in[-1,1)$ and $T>0$, let $r=r(T,x_0)$ be the (unique) positive solution to the equation*

$$\begin{aligned}&(1-x_0)(1-z_0)^2\log(1-r)+(1+x_0)(1+z_0)^2\log(1+r)-(1+2x_0z_0+z_0^2)\log r=\\ &(1-x_0)(1-z_0)^2\log(1-z_0)+(1+x_0)(1+z_0)^2\log(1+z_0)-(1+2x_0z_0+z_0^2)\log e^T z_0\end{aligned}$$

and let

$$\sigma(T,x_0)=\frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1+2x_0z_0+z_0^2}\left(\operatorname{arctanh} r(T,x_0)-\operatorname{arctanh} z_0\right).$$

If

$$T<T^* := \log\frac{(1+z_0)^2}{4z_0},$$

then $r(T,x_0)$ can be extended continuously to $x_0=1$ and we have $W_T(z_0)=\overline{W_T(z_0)}\subset\mathbb{D}$, and $W_T(z_0)$ is the closed region bounded by the two curves

$$D_\pm(z_0) := \left\{r(T,x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0\in[-1,1]\right\}.$$

Now let $T\geq T^*$ and define the two curves

$$\widetilde{D}_\pm(z_0) := \left\{r(T,x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0\in[-1,1)\right\}.$$

Here we have two cases: if T is small enough that $\widetilde{D}_+(z_0)$ and $\widetilde{D}_-(z_0)$ intersect only at $x_0=-1$, then $\overline{W_T(z_0)}$ intersects $\partial\mathbb{D}$ and $\overline{W_T(z_0)}$ is bounded by the two curves $\widetilde{D}_\pm(z_0)$ and by the part of $\partial\mathbb{D}$ between the intersection points with the curves which includes the point 1.

Otherwise, the two curves intersect on $\mathbb{R}^-$ for the first time for some $x_0=\chi\in(-1,1)$ and $\overline{W_T(z_0)}$ is the closed region bounded by $\partial\mathbb{D}$ and the two curves

$$\widehat{D}_\pm(z_0) := \left\{r(T,x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0\in[-1,\chi]\right\}.$$

In the last two cases we obtain $W_T(z_0)$ from $W_T(z_0)=\overline{W_T(z_0)}\cap\mathbb{D}$.

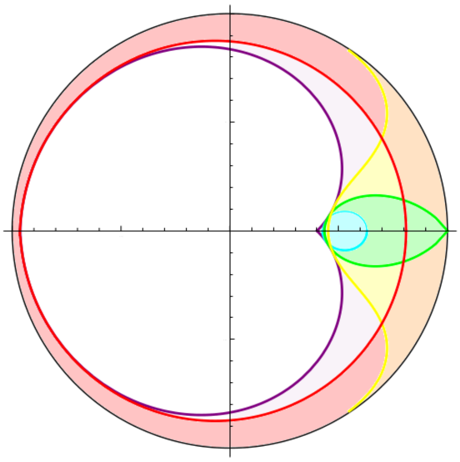

Figure 6: $W_T(0.4)$ for $T = 0.15, T^*, 0.3, 3$.

4 Proofs of Theorem 3.1 and 3.2

The proof of Theorem 3.2 is analogous to that of Theorem 1.1: we consider the inverse Loewner equation

$$\dot{w}(t) = w(t) \cdot p(t, w(t)), \quad w(0) = z_0 \in \mathbb{D}, \tag{4.1}$$

where $p(t, z)$ is a Herglotz function.
Here, a solution $t \mapsto w(t)$ may not exist for all time, i.e. there might be a $t_{max} > 0$ such that $w(t) \in \mathbb{D}$ for all $t < t_{max}$ but $|w(t)| \to 1$ for $t \uparrow t_{max}$. In this case, the (classical) solution to (4.1) ceases to exist at t_{max}. We define the reachable set

$$R'_T(z_0) = \{w(T) \,:\, w : [0, T] \to \mathbb{D} \text{ solves } (4.1)\}.$$

Note that we assume here that $w(t)$ exists up to $t = T$ and $w(T) \in \mathbb{D}$.
Then $W_T(z_0) = R'_T(z_0)$ is closed in the relative topology on $\mathbb{D}$, and we have

$$W_T(z_0) = \overline{W_T(z_0)} \cap \mathbb{D}.$$

Next we describe the boundary $\partial R'_T(z_0)$ by applying the maximum principle to (4.1). For $\mu \in \mathcal{P}$, $\lambda \in \mathbb{C}$ and $w \in \mathbb{D}$ we now have the Hamiltonian

$$H'(\mu, \lambda, w) = \lambda \cdot w \cdot p_\mu(w).$$

Since the only difference to the case $R_T(z_0)$ consists in the sign of the left hand side of the Loewner differential equation, we can use the exact same ideas as above. Equation (4.1) reduces to

$$\dot{w}(t) = w(t) \cdot \frac{\kappa(t) + w(t)}{\kappa(t) - w(t)}, \quad w(0) = z_0 \in \mathbb{D}, \tag{4.2}$$

where $\kappa : [0, T] \to \partial\mathbb{D}$ is measurable. The condition (2.8) that is satisfied by trajectories leading to boundary points now corresponds to

$$\phi = -\arg(\lambda w),$$

which means we have to solve the system of equations

$$\begin{aligned} \frac{d}{dt}|w| &= |w| \frac{1 + |w|^2 + 2|w|x}{1 - |w|^2}, \quad |w(0)| = z_0, \\ \frac{d}{dt}x &= -2 \frac{1 - x^2}{1 - |w|^2} \frac{d|w|}{dt}, \quad x(0) =: x_0 \in [-1, 1]. \end{aligned} \tag{4.3}$$

We are left with

$$x(t) = \Delta^{-1} \left(2 \operatorname{arctanh} |w(t)| - 2 \operatorname{arctanh} z_0\right),$$

where

$$\Delta(y) = \operatorname{arctanh} x_0 - \operatorname{arctanh} y,$$

and thus

$$\begin{aligned} x(t) &= \tanh\left(\operatorname{arctanh} x_0 + 2\operatorname{arctanh} z_0 - 2\operatorname{arctanh}|w(t)|\right) = \\ &= \frac{(1+|w|^2)G - 2H|w|}{(1+|w|^2)H - 2G|w|}, \end{aligned}$$

where

$$\begin{aligned} G &:= x_0 + 2z_0 + x_0 z_0^2, \\ H &:= 1 + 2x_0 z_0 + z_0^2. \end{aligned}$$

Note that, again, this last term for x is valid for any $x_0 \in [-1,1]$.
We hence arrive at

$$\frac{d|w|}{dt} = \frac{H|w|(1-|w|^2)}{H(1+|w|^2) - 2G|w|},$$

or

$$|w(t)| = \Theta^{-1}\left(-Ht + \Theta(z_0)\right)$$

with

$$\Theta(y) = (H-G)\log(1-y) - H\log y + (G+H)\log(1+y).$$

The differential equation for the argument of the optimal trajectory w reads

$$\frac{d}{dt}\arg w(t) = \pm\frac{2|w|\sqrt{H^2-G^2}}{(1+|w|^2)H - 2G|w|},$$

which means

$$\arg w(t) = \pm\frac{2\sqrt{H^2-G^2}}{H}\left(\operatorname{arctanh}|w| - \operatorname{arctanh} z_0\right).$$

We can now describe the sets $R'_T(z_0)$:
Let $x_0 \in [-1,1)$. Then $\Theta\left((0,1)\right) = (-\infty,\infty)$ and Θ is strictly decreasing. Thus there is exactly one solution $r = r(T,x_0)$ of the equation

$$\begin{aligned} &(1-x_0)(1-z_0)^2\log(1-r) + (1+x_0)(1+z_0)^2\log(1+r) - (1+2x_0z_0+z_0^2)\log r = \\ &(1-x_0)(1-z_0)^2\log(1-z_0) + (1+x_0)(1+z_0)^2\log(1+z_0) - (1+2x_0z_0+z_0^2)\log e^T z_0. \end{aligned} \tag{4.4}$$

Furthermore we define the two curves

$$\widetilde{D}_\pm(z_0) := \left\{ r(T,x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0 \in [-1,1) \right\},$$

where

$$\sigma(T,x_0) = \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1+2x_0z_0+z_0^2}\left(\operatorname{arctanh} r(T,x_0) - \operatorname{arctanh} z_0\right).$$

We take a closer look at the absolute value $r(T,x_0)$.
Firstly, the function $x_0 \mapsto r(T,x_0)$ is strictly increasing:
By solving (4.4) for T and then deriving with respect to x_0, we can calculate

$$\frac{\partial}{\partial x_0} r(T,x_0) = \frac{(1-z_0)^2 r(T,x_0)(1-r^2(T,x_0))\left(\log\left(\frac{1+r(T,x_0)}{1+z_0}\right) - \log\left(\frac{1-r(T,x_0)}{1-z_0}\right)\right)}{H(H(1+r^2(T,x_0)) - 2G\, r(T,x_0))},$$

and since the only zeros of this term lie at $r(T,x_0) = 0$, $r(T,x_0) = \pm 1$ and $r(T,x_0) = z_0$, this immediately shows that $x_0 \mapsto r(T,x_0)$ is strictly increasing in x_0 for $T > 0$.
Hence, we can define $r(T,x_0)$ also for $x_0 = 1$.

Note that for $x_0 = 1$, (4.4) simplifies to

$$2\log(1+r) - \log r = 2\log(1+z_0) - \log z_0 - T,$$

which means that the curves $D_+(z_0)$ and $D_-(z_0)$ will hit the boundary of the unit circle for the first time for

$$T = T^* := \log \frac{(1+z_0)^2}{4z_0}.$$

Next we take a closer look at the behaviour of the argument $\sigma(T, x_0)$ of the curve. We calculate

$$\frac{\partial}{\partial x_0}\sigma(T,x_0) = \frac{2(1-z_0^2)\,(\operatorname{arctanh} r(T,x_0) - \operatorname{arctanh} z_0)}{H^2}\left(\frac{2r(T,x_0)\sqrt{1-x_0^2}(1-z_0^2)^2}{(H(1+r^2(T,x_0)) - 2G\,r(T,x_0))} - \frac{G}{\sqrt{1-x_0^2}}\right).$$

Since

$$H(1+r^2(T,x_0)) - 2G\,r(T,x_0) \geq 0 \text{ for all } x_0 \in (-1,1),\ z_0 \in (0,1) \text{ and } r(T,x_0) \geq z_0,$$

the term is non-negative if and only if

$$2r(T,x_0)(1-x_0^2)(1-z_0^2)^2 \geq (HG(1+r^2(T,x_0)) - 2G^2\,r(T,x_0)),$$

or

$$H(G - 2H\cdot r(T,x_0) + G\cdot r^2(T,x_0)) \leq 0,$$

which is equivalent to

$$\frac{H-\sqrt{H^2-G^2}}{G} \leq r(T,x_0) \leq \frac{H+\sqrt{H^2-G^2}}{G} \tag{4.5}$$

The inequality to the right always holds, since

$$\frac{H+\sqrt{H^2-G^2}}{G} \quad \begin{cases} \leq 0 & \text{for } x_0 < -\frac{2z_0}{1+z_0^2}, \\ > 1 & \text{for } x_0 > -\frac{2z_0}{1+z_0^2}, \end{cases}$$

and of course

$$0 < r(T,x_0) \leq 1 \text{ for all } x_0 \in [-1,1).$$

The curves $\widetilde{D}_+(z_0)$ and $\widetilde{D}_-(z_0)$ can only intersect on $\mathbb{R}$, i.e. $\sigma(T,x_0) = k\cdot\pi$. Obviously, $\sigma(T,x_0) \geq 0$ for all x_0 so that $k \geq 0$ when the two curves intersect.

Next we show that

$$\frac{\partial}{\partial x_0}\sigma(T,x_0) > 0 \quad \text{if} \quad \sigma(T,x_0) \geq \pi. \tag{4.6}$$

We have

$$\log\left(1 + \frac{2H - 2\sqrt{H^2-G^2}}{G - H + \sqrt{H^2-G^2}}\right) \leq \frac{2H - 2\sqrt{H^2-G^2}}{G - H + \sqrt{H^2-G^2}} \leq \frac{\pi H}{\sqrt{H^2-G^2}},$$

for

$$2(H\sqrt{H^2-G^2} - H^2 + G^2) \leq \pi(H\sqrt{H^2-G^2} - H^2 + HG),$$

and thus

$$r(T,x_0) > \tanh\left(\frac{\pi H}{2(1-z_0^2)\sqrt{1-x_0^2}}\right) \geq \frac{H-\sqrt{H^2-G^2}}{G}.$$

Thus, (4.5) is satisfied in this case and $\frac{\partial}{\partial x_0}\sigma(T,x_0) > 0$.

Now we consider the first case $T < T^*$:
Here, $r(T,1) < 1$ and $\sigma(T,x_0)$ is defined also for $x_0 = 1$. Furthermore, $\sigma(T,\pm 1) = 0$, i.e. the two curves $D_+(z_0)$ and $D_-(z_0)$ intersect for $x_0 = \pm 1$ on the positive real axis. Assume that the curves intersect more than twice. As $\sigma(T,x_0) > 0$ for all $x_0 \in (-1,1)$ there must be some $\rho \in (-1,1)$ with $\sigma(T,\rho) = \pi$. This is a contradiction: the function $x_0 \mapsto \sigma(T,x_0)$ is increasing for $x_0 \in [\rho, 1]$ because of (4.6), but $\sigma(T,1) = 0$. Thus, the two curves don't intersect for $x_0 \in (-1,1)$. Consequently, the set $R_T'(z_0)$ is the closed region enclosed by $D_+(z_0) \cup D_-(z_0)$.

Next let $T = T^*$. Then $\overline{R_{T^*}'(z_0)}$ is still the closed region bounded by $D_+(z_0) \cup D_-(z_0)$, but $R_{T^*}'(z_0) = \overline{R_{T^*}'(z_0)} \setminus \{1\}$ is not closed anymore.

In passing we note that it is not difficult to show that the solution $w(t)$ of (4.2) with $\kappa(t) \equiv 1$ satisfies $\lim_{t\to T^*} w(t) = 1$ and that this case corresponds to a mapping $f \in \mathcal{S}_{T^*}$ that maps $\mathbb{D}$ onto $\mathbb{D}$ minus the slit $[z_0, 1]$.]

Now let $T > T^*$.
It is easy to see that the function Θ, which defines $r(T, x_0)$, is strictly decreasing, and that therefore, for fixed x_0, the term $r(T, x_0)$ is strictly increasing with growing T. Thus we know that we still have

$$r(T, x_0) \to 1 \text{ for } x_0 \to 1.$$

The driving function $\kappa(t) \equiv 1$ will now generate a mapping from $\mathbb{D}$ onto $\mathbb{D} \setminus [a, 1]$ with $a < z_0$. From this it is easy to deduce that

$$L(T) := \liminf_{x_0 \to 1} \sigma(T, x_0) > 0.$$

Furthermore, $L(T)$ is increasing in $T \in [T^*, \infty)$: For a point $p = e^{i\alpha} \in \partial\mathbb{D}$ the driving function $\kappa(t) \equiv -e^{i\alpha}$ has the property that $-p \cdot \frac{\kappa(t)+p}{\kappa(t)-p} = 0$. Thus, if $e^{i\alpha} \in \overline{R'_T(z_0)}$, then also $e^{i\alpha} \in \overline{R'_S(z_0)}$ for all $S \geq T$.

If T is so small that $L(T) \leq \pi$, then the curves $\widetilde{D}_\pm(z_0)$ do not intersect in $\mathbb{D}$ a second time besides $x_0 = -1$ for the same reason as in the case $T < T^*$. Here, $\overline{R'_T(z_0)}$ is the closed region which is bounded by $\widetilde{D}_+(z_0)$ and $\widetilde{D}_-(z_0)$ and the part of $\partial\mathbb{D}$ which includes the point 1.

Finally, let $L(T) > \pi$. The curves $\widetilde{D}_\pm(z_0)$ will meet at $x_0 = -1$, and then intersect again on the negative real axis before hitting $\partial\mathbb{D}$. Because of (4.6) they don't intersect more than twice provided that $T > T^*$ is small enough. Hence, in this case, $\mathbb{D} \setminus (\widetilde{D}_+(z_0) \cup \widetilde{D}_-(z_0))$ has three components, see Figure 7.

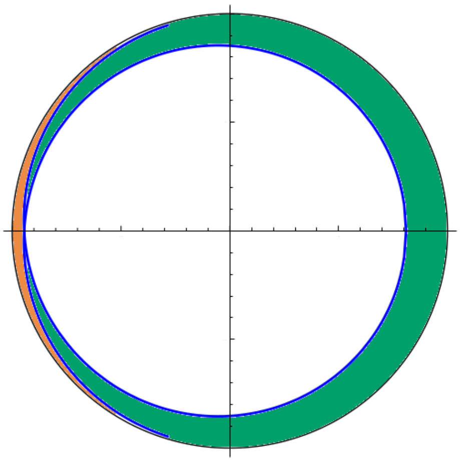

Figure 7: The decomposition of $\mathbb{D}$ by $\widetilde{D}_\pm(z_0)$

There is a simply connected component that is bounded by $\widehat{D}_+(z_0) \cup \widehat{D}_-(z_0)$ and does not touch $\partial\mathbb{D}$, and two simply connected components that do touch the boundary $\partial\mathbb{D}$. We denote by $W_T^\pm(z_0)$ the components that touch the points $+1$ (shown in green), or, respectively, -1 (shown in orange). It is clear that $\overline{R'_T(z_0)}$ has to consist of either $W_T^+(z_0)$, or $W_T^-(z_0)$, or the union of both. If it were equal to only one of the sets $W_T^\pm(z_0)$, this would imply that $\overline{R'_T(z_0)}$ is bounded away from parts of $\partial\mathbb{D}$, although $\overline{R'_t(z_0)}$, with some $t < T$, already touched these segments of $\partial\mathbb{D}$ - a contradiction. Thus, we must have $\overline{R'_T(z_0)} = \overline{W_T^+(z_0) \cup W_T^-(z_0)}$, and thus $\overline{R'_T(z_0)}$ is exactly the closed region bounded by $\partial\mathbb{D}$ and (in the interior) by $\widehat{D}_+(z_0) \cup \widehat{D}_-(z_0)$.
The same consideration applies as well for the case of more than two intersections of $\widetilde{D}_\pm(z_0)$ with $\mathbb{R}$, and for reasons of continuity, the inner boundary of $\overline{R'_T(z_0)}$ has to consists of $\widehat{D}_+(z_0) \cup \widehat{D}_-(z_0)$ in these cases, too.

We lastly show that the case where $\widetilde{D}_\pm(z_0)$ intersect for some $x_0 \in (-1, 1)$ will actually happen: For

$$x_0 = x^* := \frac{-2z_0}{1 + z_0^2},$$

(4.4) reads

$$\log(1+r)+\log(1-r)-\log r=\log(1+z_0)+\log(1-z_0)-\log z_0-T:=Y\in\mathbb{R},$$

which means

$$r=\frac{\sqrt{4+e^{2Y}}-e^Y}{2}.$$

Since $r\left(T,\frac{-2z_0}{1+z_0^2}\right)$ increases with growing T, and $r\left(T,\frac{-2z_0}{1+z_0^2}\right)\to 1$ for $T\to\infty$, it will at some point of time T become so large that

$$\operatorname{arctanh} r\left(T,\frac{-2z_0}{1+z_0^2}\right)=\frac{\pi}{2}+\operatorname{arctanh} z_0.$$

Then $\sigma(T,x^*)=2\cdot(\operatorname{arctanh} r(T,x^*)-\operatorname{arctanh} z_0)=\pi$ and consequently the curves $\widetilde{D}_\pm(z_0)$ intersect on $\mathbb{R}^-$.

This concludes the proof of The orem 3.2.

We finally prove Theorem 3.1 by applying the maximum principle to equation (4.1) in the free end time version. We have

$$\mathcal{W}_T(z_0)=\{w(T)\,:\,w:[0,\infty)\to\mathbb{D}\text{ solves (4.1)},\,T\in[0,\infty)\}.$$

If $w(t)$ is a solution with $w(T)\in\partial\mathcal{W}_T(z_0)$, then we have the same setting as above and the additional information that

$$\operatorname{Re}\,H'(\mu_t,\lambda(t),w(t))=\max_{\mu\in\mathcal{P}}\operatorname{Re}\,H'(\mu,\lambda(t),w(t))=0$$

for almost all $t\in[0,T]$, see, e.g., [Lew06], Theorem 5.18.

The optimal driving term corresponding to (2.8) thus has to fulfill

$$\cos\phi=-\frac{2|w|}{1+|w|^2},$$

which means

$$x=\frac{-2|w|}{1+|w|^2},$$

and thus (4.3) becomes

$$\frac{d}{dt}|w|=|w|\frac{1-|w|^2}{1+|w|^2},\quad |w(0)|=z_0,$$

which is equivalent to

$$|w(t)|=\frac{-1+z_0^2+\sqrt{(1-z_0^2)^2+4z_0^2e^{2t}}}{2e^tz_0}.$$

We have

$$\frac{d}{dt}\arg w(t)=\pm\frac{2|w|}{1+|w|^2},$$

which yields

$$\frac{d}{d|w|}\arg w=\pm\frac{2}{1-|w|^2},$$

or

$$\arg w=\pm2\left(\operatorname{arctanh}|w|-\operatorname{arctanh} z_0\right)=\pm(d_{\mathbb{D}}(0,|w|)-d_{\mathbb{D}}(0,z_0)).$$

Taking into account our results about the sets $W_T(z_0)$, we conclude that $\mathcal{W}(z_0)=\overline{\mathcal{W}(z_0)}\cap\mathbb{D}$ and that $\overline{\mathcal{W}(z_0)}$ is the closed region bounded by $\partial\mathbb{D}$ and the hyperbolic spirals

$$S_\pm(z_0)=\{re^{\pm i\sigma}\,:\,\sigma=d_{\mathbb{D}}(0,r)-d_{\mathbb{D}}(0,z_0),\sigma\in[0,\pi]\}.$$

This concludes the proof.

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